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Denis Efimov, Emilia Fridman

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Denis Efimov [a,b] Emilia Fridman [c]

[a] Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France

[b] ITMO University, 197101 Saint Petersburg, Russia

[c] School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

The conditions of existence of a Lyapunov-Krasovskii functional (LKF) for nonlinear input-to-state stable (ISS) neutral type systems are proposed. The system under consideration depends nonlinearly on the delayed state and the delayed state derivative, and satisfies the conditions for the existence and uniqueness of the solutions. The LKF and the system properties are defined in a Sobolev space of absolutely continuous functions with bounded derivatives.

Key words: Neutral time-delay systems, Lyapunov-Krasovskii functional, Stabilty.

1 Introduction

Stability analysis for dynamical systems constitutes an important area of research in different domains of science and technology, and especially in the control theory [11]. For generic dynamical systems, the key tool to study stability is the Lyapunov function method, which for time-delay systems have two extensions based on Lyapunov-Razumkhin functions and Lyapunov-Krasovskii functionals [5, 7, 11, 13, 14]. These approaches have also their development for systems with inputs, where one of the most popular concepts is ISS [1]. It is a well-known fact that existence of a Lyapunov function is necessary and sufficient for asymptotic stability and ISS in the case of ordinary differential equations [11]. For time-delay systems, the conditions of such an equivalence appeared rather recently (for instance, check [8, 12] for asymptotic stability, and [9, 10, 15, 16, 17, 19] for ISS case). For the class of nonlinear neutral type time-delay systems, the papers [19, 20] develop the equivalent conditions of asymptotic stability and ISS in terms of existence of LKF. In these works, the systems in the Hale's form have been analyzed, and an implicit expression of LKF has been proposed. A two steps procedure from [9] is used in [19, 20], which at the last iteration includes an infinite summation over a partition of unity, then the form of dependence of LKF on the state function is complicated and not intuitive. Studying uniform asymptotic stability these results have been extended in [2] to a different class of neutral time-delay systems (not in the Hale's form) with a coercive LKF given in an explicit form. Differently from [9, 19, 20], where the stability has been analyzed in the space $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ (the Sobolev space of continuous functions with essentially bounded derivatives), while also frequently the space $\mathbb{W}^{1,2}_{[-\tau,0]}$ is used [5] (the state derivative is square integrable), in [2] the mathematical treatment has been performed in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (the Sobolev space of continuous functions with integrable derivatives). The technical advantage of such a change consists in the established Lipschitz continuity of the solutions of the neutral type systems with locally Lipschitz continuous right-hand side in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ such a property has been proven for the systems in the Hale's form only).

The main contribution of the present work consists in the formulation of equivalent conditions of ISS for neutral type nonlinear time-delay systems with essentially bounded inputs in $\mathbb{W}^{1,1}_{[-\tau,0]}$. As in [17, 20], two types of the conditions are given: in terms of existence of an LKF, and in terms of uniform global asymptotic stability of an auxiliary system. A conference version of the paper restricted to the case of asymptotic stability was presented in [2].

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The outline of this paper is as follows. Some preliminary results are introduced in Section 2. The problem statement is given in Section 3. The main result is established in Section 4.

2 Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers, respectively, and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$.

2.1 Definitions of norms and spaces

Denote by $\mathbb{C}^n_{[a,b]}$, $[a, b] \subset \mathbb{R}$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_{[a,b]} = \sup_{s\in[a,b]} |\phi(s)|$, where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^n$.

For a Lebesgue measurable function of time $d : [a, b] \to \mathbb{R}^m$, $[a, b] \subset \mathbb{R}$, define the norm $\|d\|_{[a,b]} = \text{ess sup}_{s\in[a,b]}|d(s)|$, then $\|d\|_\infty = \text{ess sup}_{s\geq 0}|d(s)|$ and the space of d with $\|d\|_{[a,b]} < +\infty$ ($\|d\|_\infty < +\infty$) we further denote as $\mathcal{L}^m_{[a,b]}$ ($\mathcal{L}^m_\infty$).

Denote by $\mathbb{W}^{1,p}_{[a,b]}$, $p \in \mathbb{N}$ and $\mathbb{W}^{1,\infty}_{[a,b]}$ the Sobolev spaces of absolutely continuous functions $\phi : [a, b] \to \mathbb{R}^n$, $[a, b] \subset \mathbb{R}$, with bounded derivatives having the respective norms $\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} = \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds\right)^{\frac{1}{p}} < +\infty$ and $\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}} = \|\phi\|_{[a,b]} + \|\dot\phi\|_{[a,b]} < +\infty$, where $\dot\phi(s) = \frac{\partial\phi(s)}{\partial s}$ (it is a Lebesgue measurable essentially bounded function for $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$, *i.e.* $\dot\phi \in \mathcal{L}^n_{[a,b]}$) [1].

Lemma 1 *[2] For any $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$ and $p \in \mathbb{N} \cup \{+\infty\}$ the following inequalities are satisfied:*

$$\begin{aligned}\min\{1, (b-a)^{\frac{1}{p}-1}\}\|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}} &\leq \|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} \\ &\leq \max\{1, (b-a)^{\frac{1}{p}}\}\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}}.\end{aligned}$$

PROOF. From the norm definition we deduce:

$$\begin{aligned}\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} &= \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds\right)^{\frac{1}{p}} \\ &\leq \|\phi\|_{[a,b]} + \left(\int_a^b \|\dot\phi\|^p_{[a,b]} ds\right)^{\frac{1}{p}} \\ &\leq \max\{1, (b-a)^{\frac{1}{p}}\}\|\phi\|_{\mathbb{W}^{1,\infty}_{[a,b]}},\end{aligned}$$

and using integral Jensen's inequality [11]:

$$\begin{aligned}\|\phi\|_{\mathbb{W}^{1,p}_{[a,b]}} &= \|\phi\|_{[a,b]} + \left(\int_a^b |\dot\phi(s)|^p ds\right)^{\frac{1}{p}} \\ &\geq \|\phi\|_{[a,b]} + (b-a)^{\frac{1}{p}-1}\int_a^b |\dot\phi(s)| ds \\ &\geq \min\{1, (b-a)^{\frac{1}{p}-1}\}\|\phi\|_{\mathbb{W}^{1,1}_{[a,b]}}\end{aligned}$$

that was necessary to prove.

2.2 Comparison functions and their properties

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded.

Lemma 2 *[9] For any $\alpha \in \mathcal{K}$ there exists a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties: $\gamma(0) = 0$, $\gamma(s) > 0$ for all $s > 0$, and*

$$\gamma(s) \leq \alpha(s),\ |\gamma(s) - \gamma(s')| \leq |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

In addition,

$$\lim_{s\to+\infty} \gamma(s) = +\infty$$

provided that $\alpha \in \mathcal{K}_\infty$.

A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is a strictly decreasing to zero for any fixed $r \in \mathbb{R}_+$.

Lemma 3 *[21] For any $\beta \in \mathcal{KL}$ there exist $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that*

$$\beta(s, t) \leq \theta_1\left(\theta_2(s)e^{-t}\right) \quad \forall s \geq 0,\ t \geq 0.$$

2.3 Neutral systems under consideration

Consider an autonomous functional differential equation of the neutral type with inputs [14]:

$$\dot x(t) = f(x_t, \dot x_t, d(t)),\ t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in \mathbb{C}^n_{[-\tau,0]}$ is the state function, $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$, with $\dot x_t \in \mathcal{L}^n_{[-\tau,0]}$; $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}^m_\infty$. The function $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable with a constant smaller than 1, ensuring forward uniqueness and existence of the system solutions at least locally in time [14]. We assume $f(0, 0, 0) = 0$. For the initial function $x_0 \in \mathbb{C}^n_{[-\tau,0]}$ and disturbance $d \in \mathcal{L}^m_\infty$ denote a unique solution of the system (1) by $x(t, x_0, d)$, which is an absolutely continuous function defined on some time interval $[-\tau, T)$ for $T > 0$, then $x_t(x_0, d) \in \mathbb{C}^n_{[-\tau,0]}$ represents the corresponding state function, and $x_t(s, x_0, d) = x(t + s, x_0, d)$ for all $-\tau \leq s \leq 0$.

[1] In [6, 14] the norm with $p = 2$ has been only used for the state space of time-delay systems.

Given a continuous functional $V:\mathbb{R}_+\times\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\to\mathbb{R}_+$ define:

$$D^+V(t,\phi,\dot\phi,d)=\limsup_{h\to 0^+}\frac{1}{h}[V(t+h,x_h(\phi,d),\dot x_h(\phi,d)) \\ -V(t,\phi,\dot\phi)],$$

where $x_h(\phi,d)$ is a solution of the system (1) for $\phi\in\mathbb{C}^n_{[-\tau,0]}$, $\dot\phi\in\mathcal{L}^n_{[-\tau,0]}$ and $d\in\mathbb{R}^m$ is a constant.

2.4 Uniform stability

Denote $\mathcal{D}=\{d\in\mathcal{L}^m_\infty: ||d||_\infty\le 1\}$ and $\mathbb{D}=\{d\in\mathbb{R}^m: |d|\le 1\}$ in order to analyze the behavior of (1) with a bounded input d. Then in the following we summarize the well-known results [5, 13] on stability of (1).

Definition 1 *[6, 16, 18] The system* (1) *is called uniformly globally asymptotically stable (uGAS), if for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and $d\in\mathcal{D}$ there exists $\beta\in\mathcal{KL}$ such that*

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)\quad\forall t\ge 0.$$

Instead of $\mathbb{W}^{1,1}_{[-\tau,0]}$ any other space $\mathbb{W}^{1,p}_{[-\tau,0]}$ can be used in this definition for $p\in\mathbb{N}\cup\{+\infty\}$.

As it is usually assumed [6, 18], in this case

$$|x(t,x_0,d)|\le\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)\quad\forall t\ge 0.$$

Recall that in [3], under the assumption that f is Lipschitz in the second variable $\dot\phi$ with a constant smaller than 1, it is established that the latter estimate is equivalent to the stability in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$.

Definition 2 *A continuous functional $V:\mathbb{R}_+\times\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\to\mathbb{R}_+$ is called simple if $D^+V(t,\phi,\dot\phi,d)$ is independent on $\ddot\phi$.*

For instance, a locally Lipschitz functional $V:\mathbb{C}^n_{[-\tau,0]}\to\mathbb{R}_+$ is simple.

Definition 3 *[6, 16, 18] A continuous functional $V:\mathbb{R}_+\times\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\to\mathbb{R}_+$ is called LKF for the system* (1) *if it is simple, there exist $p\in\mathbb{N}\cup\{+\infty\}$, $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ and $\alpha\in\mathcal{K}$ such that V is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,p}_{[-\tau,0]}\setminus\{0\}$, and for all $t\in\mathbb{R}_+$, $d\in\mathbb{D}$ and $\phi\in\mathbb{W}^{1,p}_{[-\tau,0]}$:*

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}})\le V(t,\phi,\dot\phi)\le\alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t,\phi,\dot\phi,d)\le-\alpha(V(t,\phi,\dot\phi)).$$

Note that existence of such a LKF implies that for all $t\in\mathbb{R}_+$, $d\in\mathbb{D}$ and $\phi\in\mathbb{W}^{1,p}_{[-\tau,0]}$:

$$\alpha_1(|\phi(0)|)\le V(t,\phi,\dot\phi)\le\alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$
$$D^+V(t,\phi,\dot\phi,d)\le-\hat\alpha(|\phi(0)|),$$

where $\hat\alpha(s)=\alpha(\alpha_1(s))$ is a function from class $\mathcal{K}$, which is the standard LKF formulation used to establish asymptotic stability [5].

Theorem 1 *[5, 6] If there exists a LKF for the system* (1)*, then it is uGAS.*

There exist also some converse results to Theorem 1, see, *e.g.*, [19], which are obtained for $V:\mathbb{C}^n_{[-\tau,0]}\to\mathbb{R}_+$ and a special class of f in the Hale's form, and [9] for the background framework. Here we will use the following counterpart of Theorem 1 given in [2] (since in the sequel we have to develop some steps of the proof from [2], it is presented in the Appendix):

Theorem 2 *Let the system* (1) *be uGAS in the sense of Definition 1 and $f:\mathbb{C}^n_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\times\mathbb{R}^m\to\mathbb{R}^n$ be uniformly Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}$ (*i.e., *for any closed and bounded subset $\Upsilon\subset\mathbb{W}^{1,1}_{[-\tau,0]}$ there exists $L_\Upsilon>0$ such that*

$$|f(\phi,\dot\phi,d)-f(\varphi,\dot\varphi,d)|\le L_\Upsilon\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $\phi,\varphi\in\Upsilon$ and $d\in\mathbb{D}$). Then there exists a LKF for the system (1) *with $p=1$.*

The expression for a LKF proposed in the proof of Theorem 2, see (4) in the Appendix, has an explicit form (if only negative definiteness of $D^+V(x_0,\dot x_0,d)$ is required), and it is more simple than in [9, 19, 20], where a two step procedure for construction of LKF has been proposed.

2.5 Robust stability

The ISS property is an extension of the conventional stability paradigm to the systems with external inputs [6, 18, 23].

Definition 4 *[6, 18] The system* (1) *is called practical ISS, if for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and $d\in\mathcal{L}^m_\infty$ there exist $q\ge 0$, $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ such that for all $t\ge 0$*

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le\beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},t)+\gamma(||d||_{[0,t)})+q.$$

If $q=0$ then (1) *is called ISS.*

Definition 5 *The system* (1) *is said to possess the practical asymptotic gain (AG) property, if for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and $d\in\mathcal{L}^m_\infty$ there exist $q\ge 0$ and $\gamma\in\mathcal{K}$ such that*

$$\lim_{t\to+\infty}\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le\gamma(||d||_\infty)+q.$$

If $q=0$ then (1) *admits AG property.*

Definition 6 *The system* (1) *is said to have the practical global stability (GS) property, if for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and $d\in\mathcal{L}^m_\infty$ there exist $q\ge 0$ and $\sigma_1,\sigma_2\in\mathcal{K}$ such that for all $t\ge 0$*

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le\max\{\sigma_1(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}),\sigma_2(||d||_\infty)\}+q.$$

If $q=0$ then (1) *admits GS property.*

Again, instead of $\mathbb{W}^{1,1}_{[-\tau,0]}$ any other space $\mathbb{W}^{1,p}_{[-\tau,0]}$ can be used in these definitions for $p \in \mathbb{N} \cup \{+\infty\}$.

As it follows from the definitions above, a (practical) ISS system has (practical) AG and (practical) GS properties, and for a system in (1) described by an ordinary differential equation (*i.e.*, with $\tau = 0$) the converse implication also holds [1].

As it has been observed in [6, 18], a sufficient characterization of ISS property can be introduced for (1):

Definition 7 *A continuous functional $V : \mathbb{R}_+ \times \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called practical ISS LKF for the system (1) if it is simple and there exist $p \in \mathbb{N} \cup \{+\infty\}$, $r \geq 0$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \chi \in \mathcal{K}$ such that V is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,p}_{[-\tau,0]} \backslash \{0\}$, and for all $t \in \mathbb{R}_+$, $\phi \in \mathbb{W}^{1,p}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$:*

$$\alpha_1(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}) \leq V(t, \phi, \dot{\phi}) \leq \alpha_2(\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}}),$$

$$V(t, \phi, \dot{\phi}) \geq \max\{r, \chi(|d|)\} \Longrightarrow D^+V(t, \phi, \dot{\phi}, d) \leq -\alpha(V(t, \phi, \dot{\phi})).$$

If $r = 0$ then V is an ISS LKF.

Theorem 3 *[6] If there exists a (practical) ISS LKF for the system (1), then it is (practical) ISS with the AG $\gamma = \alpha_1^{-1} \circ \chi$.*

There exist also some converse results to Theorem 3, see *e.g.* [20], which are also obtained for $V : \mathbb{C}^n_{[-\tau,0]} \to \mathbb{R}_+$ and neutral systems in Hale's form.

3 Problem statement

The goal of this work is to propose a converse of Theorem 3 for $x_t \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and for another class of f than in [19, 20]. In particular, as in Theorem 2 the following hypothesis is accepted in the sequel:

Assumption 1 *The system (1) is ISS in the sense of Definition 4, and $f : \mathbb{C}^n_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \times \mathbb{R}^m$: for any closed and bounded subsets $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ and $D \subset \mathbb{R}^m$ there exists $L_{\Upsilon,D} > 0$ such that*

$$|f(\phi, \dot{\phi}, d) - f(\varphi, \dot{\varphi}, \delta)| \leq L_{\Upsilon,D}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d - \delta|)$$

for all $\phi, \varphi \in \Upsilon$ and $d, \delta \in D$.

Here the boundedness of a subset is understood as boundedness of the least upper bound of the norm for difference of any two elements of the subset.

An example of such a system is given by dynamics with distributed delays in the state derivatives and general delays in the state:

$$f(\phi, \dot{\phi}, d) = F(\phi, \int_{-\tau}^{0} k(s)\dot{\phi}(s)ds, d)$$

with a Lipschitz continuous on bounded sets function $F : \mathbb{C}^n_{[-\tau,0]} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and an essentially bounded kernel $k : [-\tau, 0] \to \mathbb{R}$. An important example covered by Assumption 1 includes retarded system with a pointwise delay:

$$\dot{x}(t) = F(x(t), x(t - \tau), d(t)),$$

then $f(\phi, \dot{\phi}, d) = F(\phi(0), \phi(0) - \int_{-\tau}^{0} \dot{\phi}(s)ds, d)$ and it has been observed previously [4, 5] that a LKF in $\mathbb{W}^{1,2}_{[-\tau,0]}$ for this kind of dynamics is usually more efficient than in $\mathbb{C}^n_{[-\tau,0]}$ (especially for robustness analysis).

From the relations between the norms given in Lemma 1 we observe that boundedness of x_t in $\mathbb{W}^{1,+\infty}_{[-\tau,0]}$ implies immediately a similar property for all other norms. Therefore, a stability analysis performed in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ seems to be less restrictive, which is the motivation for selection of that space in this problem formulation.

4 Main results

In this section, first, some preliminary results are established, which clarify the features and significance of the imposed assumptions, and second, a converse result is presented.

4.1 Lipschitz continuity of solutions

Under the introduced restrictions we have the following useful property for the solutions of system (1):

Proposition 1 *Let Assumption 1 be satisfied. Then in (1), for any $T > 0$, and closed and bounded subsets $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$ and $\Theta \subset \mathcal{L}^m_\infty$, there exists $M_{T,\Upsilon,\Theta} > 0$ such that*

$$\|x_t(\phi, d) - x_t(\varphi, \delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq M_{T,\Upsilon,\Theta}(\|\phi - \varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + ||d - \delta||_{[0,T)}) \quad \forall t \in [0, T]$$

for all $\phi, \varphi \in \Upsilon$ and $d, \delta \in \Theta$.

PROOF. According to Assumption 1 there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (consider also $q \geq 0$)

$$\|x_t(x_0, d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) + \gamma(||d||_\infty) + q$$

for all $t \geq 0$, for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$. For any Υ and Θ as above, $\phi, \varphi \in \Upsilon$ and $d, \delta \in \Theta$, there exists $\rho > 0$

such that $\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \rho$, $\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \rho$, $||d||_\infty \leq \rho$ and $||\delta||_\infty \leq \rho$. Hence, we have

$$\max\{\|x_t(\phi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, \|x_t(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\} \leq \beta(\rho,0) + \gamma(\rho) + q$$

and

$$|f(x_t(\phi,d),\dot{x}_t(\phi,d),d(t)) - f(x_t(\varphi,\delta),\dot{x}_t(\varphi,\delta),\delta(t))| \leq L_\rho(\|x_t(\phi,d)-x_t(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(t)-\delta(t)|)$$

for almost all $t \geq 0$, where $L_\rho > 0$ represents the Lipschitz constant of f on the set $\{\phi \in \mathbb{W}^{1,1}_{[-\tau,0]} : \|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq \beta(\rho,0)+\gamma(\rho)+q\}$ and $\{d \in \mathcal{L}^m_\infty : ||d||_\infty \leq \rho\}$ (it exists due to Assumption 1). For any $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$ we have:

$$x(t,\phi,d) = x(0,\phi,d) + \int_0^t f(x_s(\phi,d),\dot{x}_s(\phi,d),d(s))ds$$

for all $t \geq 0$, and for $\phi,\varphi \in \Upsilon$ and $d,\delta \in \Theta$:

$$\begin{aligned}|x(t,\phi,d)-x(t,\varphi,\delta)| &\leq |x(0,\phi,d)-x(0,\varphi,\delta)| \\ &+\int_0^t |f(x_s(\phi,d),\dot{x}_s(\phi,d),d(s)) - f(x_s(\varphi,\delta),\dot{x}_s(\varphi,\delta),\delta(s))|ds \\ &\leq |x(0,\phi,d)-x(0,\varphi,\delta)| \\ &+L_\rho \int_0^t \|x_s(\phi,d)-x_s(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(s)-\delta(s)|ds.\end{aligned}$$

Therefore, for $t \geq 0$:

$$\begin{aligned}\|x_t(\phi,d)-x_t(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} &= \sup_{-\tau\leq s\leq 0}|x(t+s,\phi,d)-x(t+s,\varphi,\delta)| \\ &+\int_{-\tau}^0 |\dot{x}(t+\sigma,\phi,d)-\dot{x}(t+\sigma,\varphi,\delta)|d\sigma \\ &\leq \|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \\ &+\int_0^t |f(x_\sigma(\phi,d),\dot{x}_\sigma(\phi,d),d(\sigma)) - f(x_\sigma(\varphi,\delta),\dot{x}_\sigma(\varphi,\delta),\delta(\sigma))|d\sigma \\ &+\int_{-\tau}^0 |\dot{x}(t+\sigma,\phi,d)-\dot{x}(t+\sigma,\varphi,\delta)|d\sigma \\ &\leq 2\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \\ &+2\int_0^t |f(x_\sigma(\phi,d),\dot{x}_\sigma(\phi,d),d(\sigma)) - f(x_\sigma(\varphi,\delta),\dot{x}_\sigma(\varphi,\delta),\delta(\sigma))|d\sigma \\ &\leq 2\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \\ &+2L_\rho\int_0^t \|x_\sigma(\phi,d)-x_\sigma(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |d(\sigma)-\delta(\sigma)|d\sigma \\ &\leq 2\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + 2L_\rho t||d-\delta||_{[0,t)} \\ &+2L_\rho\int_0^t \|x_\sigma(\phi,d)-x_\sigma(\varphi,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} d\sigma.\end{aligned}$$

Next, using Gronwall's Lemma [11] we obtain for all $t \in [0,T]$:

$$\|x_t(\phi,d)-x_t(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq 2(\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + L_\rho T||d-\delta||_{[0,T)})e^{2L_\rho t}.$$

Consequently, take any $T > 0$ then

$$\|x_t(\phi,d)-x_t(\varphi,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \leq M_{T,\Upsilon,\Theta}(\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + ||d-\delta||_{[0,T)}) \quad \forall t \in [0,T]$$

for $M_{T,\Upsilon,\Theta} = 2\max\{1, L_\rho T\}e^{2L_\rho T}$.

Thus, under Assumption 1, for any bounded set of initial conditions and inputs, for a compact interval of time, for the solutions of the system (1) there is a kind of local Lipschitz property with respect to the initial conditions and inputs. It is worth to highlight that the result is substantiated in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$, and a similar conclusion in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$ was obtained for the system (1) in the Hale's form [19, 20].

Remark 1 *In addition, due to Lemma 1, for any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$:*

$$|f(\phi,\dot{\phi},d) - f(\varphi,\dot{\varphi},\delta)| \leq L_{\Upsilon,D,p}(\|\phi-\varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} + |d-\delta|)$$

for all $\phi,\varphi \in \mathbb{W}^{1,p}_{[-\tau,0]}$ and $d,\delta \in D$ such that $\|\phi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \leq \min\{1,\tau^{\frac{1}{p}-1}\}\varrho$ and $\|\varphi\|_{\mathbb{W}^{1,p}_{[-\tau,0]}} \leq \min\{1,\tau^{\frac{1}{p}-1}\}\varrho$, where $L_{\Upsilon,D,p} = \max\{1,\tau^{1-\frac{1}{p}}\}L_{\Upsilon,D}$ with $L_{\Upsilon,D}$ given in Assumption 1 and $\varrho > 0$ is the corresponding norm bound on Υ, i.e. *the Lipschitz continuity of f in $\mathbb{W}^{1,p}_{[-\tau,0]}$ also follows.*

4.2 A uniformly GAS system

Further consider an auxiliary system:

$$\dot{z}(t) = f(z_t, \dot{z}_t, \gamma^{-1}(\frac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)),\ t \geq 0, \qquad (2)$$

where $z(t) \in \mathbb{R}^n$ and $z_t \in \mathbb{C}^n_{[-\tau,0]}$ is the state as before, $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, γ is the AG function given for (1) in Definition 5 (which exists for (1) under Assumption 1, and we can always assume that $\gamma \in \mathcal{K}_\infty$, then its inverse is well defined), and $\delta \in \mathcal{D}$ is a uniformly bounded input. Obviously, the origin is an equilibrium of (2) since $f(0,0,0) = 0$ in (1). In addition, for any $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $\delta \in \mathcal{D}$ the solution $z_t(z_0,\delta)$ of (2) is defined on some interval of time $[0, T^{\max}_{z_0})$ and it coincides with the solution $x_t(z_0,d)$ of (1) for $d(t) = \gamma^{-1}(\frac{1}{2}\|z_t(z_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ on this interval of time. Let us demonstrate a more strong relation between solutions of the systems (1) and (2) (as in [22], see also propositions 4.3 and 4.4 of [9]):

Proposition 2 *The following relations are true:*

i) If the system (1) *is ISS with the AG γ, then the system* (2) *is uGAS.*

ii) If the system (2) *is uGAS with the decay $\beta \in \mathcal{KL}$, then the system* (1) *is ISS with the AG $\beta(2\gamma(\cdot),0)$.*

PROOF. *i*) If the system (1) is ISS, then according to Definition 4

$$\|x_t(x_{t_0},d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_0) + \gamma(||d||_{[t_0,t)}),$$

for all $t \ge t_0$, for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and all $x_{t_0} \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $d \in \mathcal{L}^m_\infty$, $t_0 \ge 0$. Since substituting $d(t) = \gamma^{-1}(\frac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ in the system (2) we get (1), for the latter with $z_t = z_t(z_{t_0},\delta)$ and any $z_{t_0} \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $\delta \in \mathcal{D}$ we obtain:

$$\begin{aligned}
\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} &\le \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_0) \\
&+\gamma(\sup_{s\in[t_0,t)} |\gamma^{-1}(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(s)|) \\
&= \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_0) + \gamma(\sup_{s\in[t_0,t)} \gamma^{-1}(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})) \\
&= \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_0) + \frac{1}{2}\sup_{s\in[t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}
\end{aligned}$$

for all $t \in [0, T^{\max}_{z_{t_0}})$. Hence,

$$\begin{aligned}
\sup_{s\in[t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} &\le \sup_{s\in[t_0,t)} \{\beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s-t_0) \\
&+\frac{1}{2}\sup_{\sigma\in[t_0,s)} \|z_\sigma\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\} \\
&= \sup_{s\in[t_0,t)} \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s-t_0) + \frac{1}{2}\sup_{s\in[t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}.
\end{aligned}$$

Take $t_0 = 0$, then from the last inequality

$$\frac{1}{2}\sup_{s\in[0,t)} \|z_s(z_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|z_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, 0)$$

for any $t \ge 0$, all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and all $\delta \in \mathcal{D}$, which implies global uniform boundedness of the solutions of the system (2). Therefore, $T^{\max}_{z_{t_0}} = +\infty$ for all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and all $\delta \in \mathcal{D}$. Next, select $t_0 = \frac{t}{2}$:

$$\begin{aligned}
\sup_{s\in[\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} &\le \sup_{s\in[\frac{t}{2},t)} \beta(\|z_{\frac{t}{2}}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, s-\frac{t}{2}) \\
&+\frac{1}{2}\sup_{s\in[\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},
\end{aligned}$$

which for $t \to +\infty$ (*i.e.*, considering the asymptotic behavior of (2)) results in

$$\lim_{t\to+\infty}\sup_{s\in[\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \frac{1}{2}\lim_{t\to+\infty}\sup_{s\in[\frac{t}{2},t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

that can be satisfied for $\lim_{t\to+\infty} \|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 0$ only. Consequently, the system (2) is uniformly converging to the origin, the uGAS property has been established.

ii) Conversely, let (2) be uGAS, then according to Definition 1:

$$\|z_t(z_{t_0},\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|z_{t_0}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_0), \quad \forall t \ge t_0,$$

for some $\beta \in \mathcal{KL}$ and all $z_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, $\delta \in \mathcal{D}$, $t_0 \ge 0$. Consider the system (1), take any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d \in \mathcal{L}^m_\infty$, assume that $||d||_\infty \le \gamma^{-1}(\frac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [0,t_1)$ with some (possibly infinite) $t_1 \ge 0$, then there is $\delta \in \mathcal{D}$ such that $d(t) = \gamma^{-1}(\frac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t)$ for all $t \in [0,t_1)$ and, hence, using the properties of (2) we obtain:

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t), \quad \forall t \in [0,t_1).$$

Next, assume that $||d||_\infty > \gamma^{-1}(\frac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [t_1,t_2)$ with some $t_2 > t_1$ (obviously, always $t_1+t_2 > 0$ for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$), hence,

$$\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le 2\gamma(||d||_\infty), \quad \forall t \in [t_1,t_2).$$

Finally, let again $||d||_\infty \le \gamma^{-1}(\frac{1}{2}\|x_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$ for $t \in [t_2,t_3)$ with some $t_3 > t_2$, similarly

$$\begin{aligned}
\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} &\le \beta(\|x_{t_2}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t-t_2) \\
&\le \beta(2\gamma(||d||_\infty), 0), \quad \forall t \in [t_2,t_3)
\end{aligned}$$

since in this case $\|x_{t_2}\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} = 2\gamma(||d||_\infty)$ by construction. Next, all these steps can be repeated iteratively if necessary (they cover all possible scenarios), and it is clear that for all $t \ge 0$

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \beta(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}, t) + \beta(2\gamma(||d||_\infty), 0),$$

since by definition $\beta(s,0) \ge s$ for all $s \ge 0$, which implies ISS property of (1).

4.3 Converse design of LKF

For brevity of exposition the analysis is presented in the space $\mathbb{W}^{1,1}_{[-\tau,0]}$ (we need the result of Proposition 1 formulated in this space) and for ISS case only.

Theorem 4 *Let Assumption 1 be satisfied, then there exists an ISS LKF for the system* (1).

PROOF. Under the introduced hypothesis and Lemma 3 there are $\theta_1, \theta_2, \theta_3 \in \mathcal{K}_\infty$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \theta_1\left(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\right) + \theta_3(||d||_\infty)$$

for all $t \ge 0$, for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$. Recalling Lemma 2, there exists a continuous, positive definite and radially unbounded function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties:

$$\psi(s) \le \theta_3^{-1}(s), \ |\psi(s) - \psi(s')| \le |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

Following the arguments of Proposition (2), the system (that is a variant of (2))

$$\dot{z}(t) = F(z_t, \dot{z}_t, \delta(t)), \ t \ge 0, \tag{3}$$

where $z(t) \in \mathbb{R}^n$, $z_t \in \mathbb{C}^n_{[-\tau,0]}$, $\delta \in \mathcal{D}$ and $F(z_t, \dot{z}_t, \delta(t)) = f(z_t, \dot{z}_t, \psi(\frac{1}{2}\|z_t\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta(t))$, is uGAS. Indeed,

$$\begin{aligned}\theta_3(\sup_{s\in[t_0,t)} \psi(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})) &\le \theta_3(\sup_{s\in[t_0,t)} \theta_3^{-1}(\frac{1}{2}\|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}))\\ &= \frac{1}{2}\sup_{s\in[t_0,t)} \|z_s\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\end{aligned}$$

as for (2) (and all other arguments of Proposition (2) stay unchanged). Now, take any closed and bounded subset $\Upsilon \subset \mathbb{W}^{1,1}_{[-\tau,0]}$, according to Assumption 1 there exists $L_{\Upsilon,\mathbb{D}} > 0$ such that

$$\begin{aligned}|F(\phi,\dot{\phi},\delta) - F(\varphi,\dot{\varphi},\delta)| &= |f(\phi,\dot{\phi},\psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta)\\ &\quad -f(\varphi,\dot{\varphi},\psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta)|\\ &\le L_{\Upsilon,\mathbb{D}}(\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |\psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta\\ &\quad -\psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta|)\\ &\le L_{\Upsilon,\mathbb{D}}(\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + |\psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\\ &\quad -\psi(\frac{1}{2}\|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})|)\\ &\le L_{\Upsilon,\mathbb{D}}(\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} + \frac{1}{2}|\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} - \|\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}|)\\ &\le \frac{3}{2}L_{\Upsilon,\mathbb{D}}\|\phi-\varphi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\end{aligned}$$

for all $\phi, \varphi \in \Upsilon$ and $\delta \in \mathbb{D}$, where the Lipschitz properties of ψ and $\|\cdot\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ were used on the last two steps. Therefore, F is uniformly Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}$, then all conditions of Theorem 2 are satisfied and the system (3) admits a LKF V, which is Lipschitz continuous on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]} \setminus \{0\}$, and for all $\delta \in \mathbb{D}$ and $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$:

$$\begin{aligned}\alpha_1(\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \le V(\phi,\dot{\phi}) &\le \alpha_2(\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}),\\ D^+V(\phi,\dot{\phi},\delta) &\le -\alpha(V(\phi,\dot{\phi}))\end{aligned}$$

for some $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$. Returning to the system (1), for any $\phi \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$, let $\underline{\psi}(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \ge |d|$, where $\underline{\psi}(s) = \frac{s}{1+s}\inf_{\sigma\ge s}\psi(\sigma)$ is a function from class $\mathcal{K}_\infty$, then $d = \underline{\psi}(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta' = \psi(\frac{1}{2}\|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\delta$ for some $\delta', \delta \in \mathbb{D}$, hence, the following implication holds in (1):

$$\begin{aligned}V(\phi,\dot{\phi}) \ge \chi(|d|) &\Rightarrow \|\phi\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \ge 2\underline{\psi}^{-1}(|d|)\\ &\Rightarrow D^+V(\phi,\dot{\phi},d) \le -\alpha(V(\phi,\dot{\phi})),\end{aligned}$$

where $\chi(s) = \alpha_2(2\underline{\psi}^{-1}(s))$, which implies that V is an ISS LKF for (1).

The obtained result shows that the existence of a coercive ISS LKF is necessary and sufficient for ISS of the system (1) in $\mathbb{W}^{1,1}_{[-\tau,0]}$ provided that the Lipschitz continuity property stated in the Assumption 1 is verified. The proposed conditions establish a direct relation between a LKF for the auxiliary system (3) and the required ISS LKF of (1).

5 Conclusions

The problem of existence of an ISS LKF for nonlinear neutral type time-delay systems is solved considering the ISS property in $\mathbb{W}^{1,1}_{[-\tau,0]}$ space. It is shown that Lipschitz property of f defined in such a space can be transformed to the same property of the solutions of (1), and relations between ISS and uGAS properties of (1) and an auxiliary system (2) are established.

Appendix

Under the introduced hypothesis in Theorem 2, and Lemma 3, there are $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that

$$\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}} \le \theta_1\left(\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\right) \quad \forall t \ge 0$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ and $d \in \mathcal{D}$, then by recalling Lemma 2, there exists a continuous, positive definite and radially unbounded function $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$ admitting the following properties:

$$\varsigma(s) \le \theta_1^{-1}(s),\ |\varsigma(s) - \varsigma(s')| \le |s - s'| \quad \forall s, s' \in \mathbb{R}_+.$$

Now, for any $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$ select

$$V(x_0,\dot{x}_0) = \sup_{t\ge0, d\in\mathcal{D}}\{\varsigma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\} \tag{4}$$

with $\frac{\kappa_2}{1+\kappa_2} < \kappa_1 < \kappa_2 < +\infty$. Then

$$\begin{aligned}V(x_0,\dot{x}_0) &\le \sup_{t\ge0, d\in\mathcal{D}}\{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\frac{\kappa_1+t}{\kappa_2+t}\}\\ &\le \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\end{aligned}$$

and

$$\begin{aligned}V(x_0,\dot{x}_0) &\ge \sup_{t\ge0, d\in\mathcal{D}}\{\varsigma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2}\}\\ &\ge \varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2}\end{aligned}$$

since under introduced restrictions on κ_1 and κ_2 the functions $\frac{\kappa_1+t}{\kappa_2+t}$ and $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ are strictly increasing and decreasing, respectively. Define $\underline{\varsigma}(s) = \frac{s}{1+s}\inf_{\sigma\ge s}\varsigma(\sigma)$, which is a function from class $\mathcal{K}_\infty$, then

$$\frac{\kappa_1}{\kappa_2}\underline{\varsigma}(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}) \le V(x_0,\dot{x}_0) \le \frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

for all $x_0 \in \mathbb{W}^{1,1}_{[-\tau,0]}$, and the LKF $V(x_0,\dot{x}_0)$ is coercive admitting lower and upper bounds in terms of functions from the class $\mathcal{K}_\infty$ with respect to $\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$ as desired.

From the decreasing of $e^{-t}\frac{\kappa_1+t}{\kappa_2+t}$ it also follows that there exists $T^{x_0}>0$ such that

$$V(x_0,\dot{x}_0)=\sup_{0\le t\le T^{x_0},d\in\mathcal{D}}\{\varsigma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\},$$

and since

$$\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1}{\kappa_2}\le V(x_0,\dot{x}_0)$$
$$\le\sup_{0\le t\le T^{x_0},d\in\mathcal{D}}\{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})e^{-t}\},$$

then by the definition of T^{x_0}, it has an upper estimate:

$$T^{x_0}\le\ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}{\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})}\right].$$

For any $0<r<R<+\infty$ define a set

$$\Omega_{r,R}=\{x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}:r\le\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le R\},$$

consequently, there exists a finite $T^{r,R}=\sup_{x_0\in\Omega_{r,R}}T^{x_0}$, *i.e.* $T^{r,R}=\ln\left[\frac{\kappa_2}{\kappa_1}\frac{\theta_2(R)}{\varsigma(r)}\right]$.

Let us check the Lipschitz continuity of V on any bounded and closed subset in $\mathbb{W}^{1,1}_{[-\tau,0]}\setminus\{0\}$ (here by $\{0\}$ we understand $x\in\mathbb{W}^{1,1}_{[-\tau,0]}$ with $\|x\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}=0$). For any $x_0,x_1\in\mathbb{W}^{1,1}_{[-\tau,0]}\setminus\{0\}$ denote $T^{x_0,x_1}=\max\{T^{x_0},T^{x_1}\}$ and consider:

$$|V(x_1,\dot{x}_1)-V(x_0,\dot{x}_0)|=|\sup_{0\le t\le T^{x_1},d\in\mathcal{D}}\{\varsigma(\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\}$$
$$-\sup_{0\le t\le T^{x_0},d\in\mathcal{D}}\{\varsigma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+t}{\kappa_2+t}\}|$$
$$\le\sup_{0\le t\le T^{x_0,x_1},d\in\mathcal{D}}\left|[\varsigma(\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})-\varsigma(\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})]\frac{\kappa_1+t}{\kappa_2+t}\right|$$
$$\le\sup_{0\le t\le T^{x_0,x_1},d\in\mathcal{D}}\left|\|x_t(x_1,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}-\|x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\right|$$
$$\le\sup_{0\le t\le T^{x_0,x_1},d\in\mathcal{D}}\|x_t(x_1,d)-x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}},$$

where on the last step and the step before the Lipschitz properties of the norm $\|\cdot\|_{\mathbb{W}^{1,\infty}_{[-\tau,0]}}$ and the function ς have been utilized, respectively. For any $x_0,x_1\in\mathbb{W}^{1,1}_{[-\tau,0]}\setminus\{0\}$ there exist $0<r<R<+\infty$ such that $x_0,x_1\in\Omega_{r,R}$, then $T^{x_0,x_1}\le T^{r,R}$ and using Proposition 1 in [2] (see also Proposition 1) there exists $M_{T^{r,R},\Omega_{r,R}}>0$ such that

$$\|x_t(x_1,d)-x_t(x_0,d)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}\le M_{T^{r,R},\Omega_{r,R}}\|x_1-x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $t\in[0,T^{r,R}]$ and all $d\in\mathcal{D}$, hence,

$$|V(x_1,\dot{x}_1)-V(x_0,\dot{x}_0)|\le M_{T^{r,R},\Omega_{r,R}}\|x_1-x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}}$$

for all $x_0,x_1\in\Omega_{r,R}$ and any $0<r<R<+\infty$, which implies the required Lipschitz continuity of V on bounded sets in $\mathbb{W}^{1,1}_{[-\tau,0]}\setminus\{0\}$. The continuity at the origin follows from the upper estimate:

$$V(x_0,\dot{x}_0)\le\frac{\kappa_1}{\kappa_2}\theta_2(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

that is satisfied for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$.

Finally, let us check the decreasing of the LKF V on the trajectories of the system (1) for $t>0$:

$$V(x_t(x_0,d),\dot{x}_t(x_0,d))$$
$$=\sup_{\sigma\ge0,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_t(x_0,d),\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}$$
$$=\sup_{\sigma\ge t,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma-t}{\kappa_2+\sigma-t}\}$$
$$<\sup_{\sigma\ge t,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}$$
$$\le\sup_{\sigma\ge0,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}=V(x_0,\dot{x}_0),$$

thus, V is strictly decreasing along the trajectories of (1) for all $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d\in\mathcal{D}$. Moreover (this part is absent in [2]),

$$V(x_t(x_0,d),\dot{x}_t(x_0,d))-V(x_0,\dot{x}_0)$$
$$=\sup_{\sigma\in[0,T^{x_t(x_0,d)}],\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_t(x_0,d),\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}$$
$$-\sup_{\sigma\ge0,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}$$
$$=\sup_{\sigma\in[0,T^{x_t(x_0,d)}],\delta\in\mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$
$$\times\left(\frac{\kappa_1+t+\sigma}{\kappa_2+t+\sigma}+\frac{\kappa_1+\sigma}{\kappa_2+\sigma}-\frac{\kappa_1+t+\sigma}{\kappa_2+t+\sigma}\right)\}$$
$$-\sup_{\sigma\ge0,\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_1+\sigma}{\kappa_2+\sigma}\}$$
$$\le\sup_{\sigma\in[0,T^{x_t(x_0,d)}],\delta\in\mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$
$$\times\left(\frac{\kappa_1+\sigma}{\kappa_2+\sigma}-\frac{\kappa_1+t+\sigma}{\kappa_2+t+\sigma}\right)\},$$

and applying the Mean Value Theorem we obtain:

$$V(x_t(x_0,d),\dot{x}_t(x_0,d))-V(x_0,\dot{x}_0)$$
$$\le-\sup_{\sigma\in[0,T^{x_t(x_0,d)}],\delta\in\mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})t$$
$$\times\frac{\kappa_2-\kappa_1}{(\kappa_2+\sigma+\delta t)^2}\}$$

for some $\delta t\in(0,t)$. Finally,

$$D^+V(x_0,\dot{x}_0,d)=\limsup_{t\to0^+}\frac{1}{t}[V(x_t(x_0,d),\dot{x}_t(x_0,d))-V(x_0,\dot{x}_0)]$$
$$\le-\limsup_{t\to0^+}\sup_{\sigma\in[0,T^{x_t(x_0,d)}],\delta\in\mathcal{D}}\{\varsigma(\|x_{t+\sigma}(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_2-\kappa_1}{(\kappa_2+\sigma+\delta t)^2}\}$$
$$=-\sup_{\sigma\in[0,T^{x_0}],\delta\in\mathcal{D}}\{\varsigma(\|x_\sigma(x_0,\delta)\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})\frac{\kappa_2-\kappa_1}{(\kappa_2+\sigma)^2}\}$$
$$\le-\frac{\kappa_2-\kappa_1}{(\kappa_2+T^{x_0})^2}\varsigma(\|x_0\|_{\mathbb{W}^{1,1}_{[-\tau,0]}})$$

for any $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d\in\mathbb{D}$, which is a negative definite function of V, but probably globally bounded. Now defining a new LKF $\tilde{V}(x_0,\dot{x}_0)=\int_0^{V(x_0,\dot{x}_0)}\rho(s)ds$ for a suitably defined function $\rho\in\mathcal{K}_\infty$ it is possible to obtain another LKF with a negative definite and properly unbounded derivative $D^+\tilde{V}(x_0,\dot{x}_0,d)$ for any $x_0\in\mathbb{W}^{1,1}_{[-\tau,0]}$ and any $d\in\mathbb{D}$, as desired.

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